On Societies Choosing Social Outcomes, and their Memberships: Internal Stability and Consistency*

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Abstract: We consider a society whose members have to choose not only an outcome from a given set of outcomes but also the subset of agents that will remain members of the society. We study the extensions of approval voting, plurality voting, scoring methods and the Condorcet winner to our setting from the point of view of their internal stability and consistency properties.

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1 Introduction

Classical social choice studies problems where a fixed set of agents have to choose an outcome from a given set of outcomes, and agents have preferences only over this set. However, there are settings where, depending on the chosen outcome, some agents might want to leave the society; and this, in turn, might be perceived by some agents that were initially willing to remain in the society as negative, and now they might also want to leave. Membership in a political party may depend on the positions that the party takes on issues like the death penalty, abortion or the possibility of allowing a region of a country to become independent. A professor in a department may start looking for a position elsewhere if he considers that the recruitment of the department has not being satisfactory to his standards; and this, in turn might trigger further exits. To be able to deal with such situations the classical social choice model has to be modified to include explicitly the possibility that initial members of the society may leave it as the consequence of the chosen outcome and hence, preferences have to be extended to order pairs formed by the final society and the chosen outcome.

There is a large literature that has already considered explicitly the dependence of the final society on the choices made by the initial society.[1] Barberà, Maschler and Shalev (2001), Barberà and Perea (2002), and Berga, Bergantiños, Massó and Neme (2004, 2006, 2007) study alternative models in terms of the voting methods used to choose the outcome and the timing under which members reconsider their membership. In this note (as we also do in the companion paper Bergantiños, Massó, and Neme (2016)) we look at the general setting without being specific about the two issues. We do that by considering that the set of alternatives are all pairs formed by a subset of the original society (an element in 2^N, the subset of agents that will remain in the society) and an outcome in X. Then, we assume that agents' preferences are defined over the set of alternatives $2^N \times X$ and satisfy two natural requirements. First, each agent has strict preferences between any two alternatives, provided he belongs to at least one of the two corresponding societies. Second, each agent is indifferent between any two alternatives, provided he is not a member of any of the two corresponding societies; namely, agents that do not belong to the final society do not care about neither its composition nor the chosen outcome.

We consider rules that operate on this restricted domain of preference profiles by selecting, for each profile, an alternative (a final society and an outcome). In Bergantiños, Massó, and

[1] See for instance Roberts (1999) for problems related to club formation and Sobel (2000) for the declining of standards in societies that chose their members.

Neme (2016) we characterize the class of strategy-proof, unanimous and non-bossy rules as the family of all serial dictator rules.

For applications where the profile is common knowledge (and hence, the strategic revelation of agents' preferences is not an issue) we focus on internal stable and consistent rules.[2] Internal stability says that nobody can force an agent to remain in the society if the agent does not want to do so. This is a minimal requirement of individual rationality, and it is a desirable property whenever membership is voluntary. A rule is consistent if the following property holds. Apply the rule to a given profile and consider the new problem where the new society is formed by the subset of agents chosen at the original profile. A consistent rule has to choose, at the subprofile of preferences of the agents that remain in the society, the same alternative. Thus, a consistent rule does not have to be reapplied after an alternative has been chosen. Internal stability and consistency are desirable if we want to interpret the alternative chosen by the rule as being the final one, in a double sense. Members of the final society want to stay and if the rule would be applied again to the final society it would chose the same final society and the same outcome, so there is no need to do so.

We adapt well-known voting methods to our setting with the goal of making them either internally stable or consistent, or both. We show that plurality voting and scoring methods do not satisfy consistency. However, approval voting not only satisfies internal stability and consistency but it also satisfies efficiency and neutrality. Finally, we show that the Condorcet winner is internal stable, consistent, efficient, neutral and anonymous at those profiles where an alternative beats all other alternatives by majority voting (namely, whenever it is a well-defined rule).

The paper is organized as follows. In Section 2 we describe the model. Section 3 contains the definitions of the properties of rules that we are interested in. Section 4 contains the analysis of well-known rules from the point of view of their internal stability and consistency properties.

2 Preliminaries

This section follows closely Bergantiños, Massó, and Neme (2016). Let $N = \{1, \ldots, n\}$, with $n \geq 2$, be the set of *agents* who have to chose an outcome from a given set X of possible

[2] For the study of consistent rules in other social choice settings see, for instance, Sasaki and Toda (1992), Thomson (1994, 2007), Özkal-Sanver (2013), Nizamogullari and Özkal-Sanver (2014, 2015) and Bergantiños, Massó and Neme (2015).

outcomes. We are interested in situations where some agents may not be part of the final society, perhaps as the consequence of the chosen outcome. To model such situations, let $A = 2^N \times X$ be the set of *alternatives* and assume that each $i \in N$ has preferences over A. Observe that for all $x \in X$, $(\varnothing, x) \in A$; so we are admitting the possibility that the final society does not have any member. We will often use the notation a for a generic alternative $(S, x) \in A$; *i.e.*, $a \equiv (S, x)$, $a' \equiv (S', x')$, and so on. Let R_i denote i's (weak) *preference* over A, where for any pair $a, a' \in A$, aR_ia' means that i considers a to be at least as good as a'. Let P_i and I_i denote the strict and indifference relations over A induced by R_i, respectively; namely, for any pair $a, a' \in A$, aP_ia' if and only if aR_ia' and $\neg a'R_ia$, and aI_ia' if and only if aR_ia' and $a'R_ia$. We assume that each i does not care about all alternatives at which i does not belong to their corresponding final societies. Besides i is not indifferent between any pair of alternatives at which i belongs to at least one of the two corresponding final societies. Namely, we assume that i's preferences R_i satisfy the following two properties: for all $S, T \in 2^N$ and $x, y \in X$,

(P.1) if $i \notin S \cup T$ then $(S, x)\, I_i\, (T, y)$; and

(P.2) if $i \in S \cup T$ and $(S, x) \neq (T, y)$ then either $(S, x)\, P_i\, (T, y)$ or $(T, y)\, P_i\, (S, x)$.

The fact that agents' preferences satisfy (P.1) is the reason why our model cannot mechanically be embedded into the classical model. A specific analysis is required, partly because properties like internal stability and consistency become specially meaningful under this domain restriction. We see property (P.1) as being a natural assumption for our setting, and it is a critical requirement for our results to hold. Let $\mathcal{R}_i$ be the set of preferences of i satisfying (P.1) and (P.2), and let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ be the set of (preference) *profiles*.

We denote the subset of alternatives with the property that i is not a member of the corresponding final society by $[\varnothing]_i = \{(S, x) \in A \mid i \notin S\}$. By (P.1), i is indifferent among them; *i.e.*,

$$[\varnothing]_i = \{a \in A \mid aI_i\,(\varnothing, x) \text{ for some } x \in X\}.$$

By (P.1), $(\varnothing, x)I_i(\varnothing, y)$ for all $x, y \in X$ and $[\varnothing]_i$ can be seen as the indifference class generated by the empty society. Observe that $[\varnothing]_i$ may be at the top of i's preferences. With an abuse of notation we often treat, when listing a preference ordering, the indifference class $[\varnothing]_i$ as if it were an alternative; for instance, given R_i and $a \in A$ we write $aR_i[\varnothing]_i$ to represent that aR_ia' for all $a' \in [\varnothing]_i$.

The *top* of R_i, denoted by $\tau(R_i)$, is the set of all best alternatives according to R_i; namely,

$$\tau(R_i) = \{a \in A \mid aR_ia' \text{ for all } a' \in A\}.$$

A *rule* is a social choice function $f : \mathcal{R} \to A$ selecting, for each profile $R \in \mathcal{R}$, an alternative $f(R) \in A$. To be explicit about the two components of the alternative chosen by f at R, we will often write $f(R)$ as $(f_N(R), f_X(R))$, where $f_N(R) \in 2^N$ and $f_X(R) \in X$.

To clarify the model, we relate it with the two examples used in the introduction. The set of agents N corresponds to the initial members of the political party, the set of outcomes X to the set of choices that the political party has to make and the set S, if the chosen alternative is (S, x), to the set of final members of the party that stay after it supports outcome x. Similarly, N corresponds to the set of professors in the department, the set of outcomes X to all subsets of candidates and the set S, if the chosen alternative is (S, x), to the set of professors who remain in the department after the subset of candidates x has been hired.

3 Properties of rules

In this section we present several properties that a rule may satisfy. The first two impose conditions at each profile.

A rule is efficient if it always selects a Pareto optimal allocation.

Efficiency For each $R \in \mathcal{R}$ there is no $a \in A$ with the property that $aR_if(R)$ for all $i \in N$ and $aP_jf(R)$ for some $j \in N$.

The next property is related to the stability of a rule, and it captures the idea that agents are able to exit a society at their free will. Internal stability says that no agent belonging to the final society would prefer to leave it.

Internal stability For all $R \in \mathcal{R}$ and all $i \in f_N(R)$, $f(R)\, P_i\, [\varnothing]_i$.

It is immediate to see that internal stability is indeed equivalent to the requirement of individual rationality (for *all* agents); *i.e.*, for all R and all i, $f(R)R_i[\varnothing]_i$. Individual rationality implies internal stability by their definitions and (P.2). Assume f is internally stable and let R be arbitrary. If $i \in f_N(R)$ then $f(R)P_i[\varnothing]_i$. If $i \notin f_N(R)$ then, by (P.1), $f(R)I_ia$ for any $a \in [\varnothing]_i$. Thus, for all i, $f(R)R_i[\varnothing]_i$. It is easy to see that serial dictator rules, as defined in Bergantiños, Massó and Neme (2016), are not internally stable.

The next three properties impose conditions by comparing the alternatives chosen by the rule at two different profiles. A rule is anonymous if the names of the agents are not relevant to select the alternative. To define it formally, let $\pi : N \to \{1, \ldots, n\}$ be an ordering of N (*i.e.*, a one-to-one mapping). Given $i \in N$, $\pi(i)$ is the agent assigned to i after applying π to N. The set of all orderings $\pi : N \to \{1, \ldots, n\}$ will be denoted by Π. Given $S \in 2^N$ and $\pi \in N$

we denote by $\pi(S)$ the subset of agents associated to S by π; namely, $\pi(S) = \{i \in N \mid \pi(j) = i$ for some $j \in S\}$. Given $R \in \mathcal{R}$ and $\pi \in \Pi$ we denote by R^{π} the new profile where, for all $i \in N$, agent $\pi(i)$ has the preference obtained from R_i after replacing each (S, x) by $(\pi(S), x)$.

Anonymity For all $R \in \mathcal{R}$ and all $\pi \in \Pi$, $f(R^{\pi}) = (\pi(f_N(R)), f_X(R))$.

A rule is neutral if the names of the outcomes do not play any role in selecting the social alternative. To define it formally, let $\sigma : X \to X$ be a permutation of X. Given $x \in X$, $\sigma(x)$ is the outcome assigned to x after applying σ to X. The set of all permutations $\sigma : X \to X$ will be denoted by Σ. Let $Y \subseteq X$ be non-empty and $\sigma \in \Sigma$. Denote by $\sigma(Y)$ the subset of outcomes associated to Y by σ; namely, $\sigma(Y) = \{x \in X \mid \sigma(y) = x$ for some $y \in Y\}$. Given $R \in \mathcal{R}$ and $\sigma \in \Sigma$ we denote by R^{σ} the profile where, for all $i \in N$, the preference R_i^{σ} is obtained from R_i after replacing each (S, x) by $(S, \sigma(x))$.

Neutrality For all $R \in \mathcal{R}$ and all $\sigma \in \Sigma$, $f(R^{\sigma}) = (f_N(R), \sigma(f_X(R)))$.

A rule is consistent if the following requirement holds. Apply the rule to a given profile and consider the subset of agents that are members of the final society. Construct the new subprofile of preferences restricted to this new set of chosen agents. Then, the rule does not require to modify the chosen alternative because if it were applied to the new subprofile the new alternative would coincide with the alternative chosen at the original profile. To define the property formally, we first need an additional notation. Given $R \in \mathcal{R}$ and $S \subset N$, denote by $R_{|S} = ((R_{|S})_i)_{i \in S}$ the restriction of R to $2^S \times X$. Namely, given $i \in S$, $T \cup T' \subset S$ and $x, y \in X$, $(T, x) \left(R_{|S}\right)_i (T', y)$ if and only if $(T, x) \, R_i \, (T', y)$. Second, we specify how a given rule f can be applied to a subprofile by considering it as it were a family of rules, one for each non-empty subset of N. Given $S \in 2^N \backslash \{\varnothing\}$ denote by $\mathcal{R}^S$ the set of subprofiles $R_{|S} = ((R_{|S})_i)_{i \in S}$. Thus, a rule f can be identified with the collection $\{f^S\}_{S \in 2^N \backslash \{\varnothing\}}$ of rules where for each $S \in 2^N \backslash \{\varnothing\}$, $f^S : \mathcal{R}^S \to 2^S \times X$. We often omit the superscript S and write $f(R_{|S})$.

Consistency For all $R \in \mathcal{R}$, $f(R) = f\left(R_{|f_N(R)}\right)$ whenever $f_N(R) \neq \varnothing$.

In contrast with the standard notion, our consistency property requires to re-apply the rule only to the (non-empty) set of agents that has been selected at the original profile. We think that this is the relevant consistency notion because the new composition of the society is not just a hypothetical circumstance, it is a fact. And indeed, the new set of agents might be willing to reconsider their membership and the chosen outcome; particularly because, in the choice of the later, preferences of members that are not anymore in the society may have played a relevant role. Consistency says that the original choice, if re-evaluated by the new society by means of the same rule, will continue to be chosen.

We say that a rule satisfies any of the above properties at R if the condition defining the property holds at R.

4 Internally stable and consistent rules

In Bergantiños, Massó and Neme (2016) we characterize the class of all strategy-proof, unanimous and non-bossy rules as the family of serial dictator rules. Here, we consider situations where the strategic manipulation in the preference revelation game is not an issue and we will look for internally stable and consistent rules. To do so, we first ask whether three of the most prominent procedures in classical social choice satisfy them. Recall that in the classical setting the goal is to select an outcome, from a given set X, taking into account (partially or fully) the *strict* preferences of agents over X. The rules we consider are:

1. Approval voting. Each $i \in N$ votes for a subset X_i of X. For each outcome $x \in X$, compute the number of votes received by x; namely, $|\{i \in N \mid x \in X_i\}|$. The outcome with more votes is selected. A tie-breaking rule should be applied whenever two or more outcomes obtain the largest number of votes. Note that approval voting is not a rule because i's vote X_i is not completely determined by P_i.

2. Plurality voting. Each $i \in N$ votes for an outcome $x_i \in X$. The outcome with more votes is selected. A tie-breaking rule should be applied whenever two or more outcomes obtain the largest number of votes.

3. Scoring methods. Each $i \in N$ strictly ranks all outcomes. Assign to each outcome a pre-established decreasing number of points depending on its position in i's ranking.[3] Compute the sum of the points obtained by each outcome. Select the one with more points. A tie-breaking rule should be applied whenever two or more outcomes obtain the largest number of points.

We tentatively adapt the three voting methods to our setting to deal with the indifferences generated by (P.1) and to define approval voting as a proper rule.

1. Approval voting. Each $i \in N$ votes for all $a \in A$ such that $aP_i [\varnothing]_i$ (if any).

2. Plurality voting. Each $i \in N$ votes for his top alternative $\tau(R_i)$. If $\tau(R_i) = [\varnothing]_i$ then i votes for all $a \in [\varnothing]_i$.

[3]The Borda rule is the scoring method when the points are the integers $|X| - 1, \ldots, 0$.

3. Scoring methods. For each $i \in N$, assign a pre-established decreasing number of points to each outcome depending on its position in i's ranking but considering $[\varnothing]_i$ as a single alternative. For each $(S,x) \in A$ and each $i \in N \backslash S$, assign to (S,x) the score obtained by $[\varnothing]_i$.

Example 1 below shows that none of these extensions satisfy internal stability.

Example 1 Assume $n \geq 3$ and fix $x \in X$. Let $R \in \mathcal{R}$ be any profile such that $\tau(R_1) = [\varnothing]_1$ and for all $i \in N \backslash \{1\}$, $\tau(R_i) = (N,x)$ and $[\varnothing]_i P_i (S,y)$ for all $S \neq N$, $i \in S$ and $y \in X$. Then, the three adapted voting methods choose (N,x) at R. Nevertheless, (N,x) is not internally stable because agent 1 prefers to leave the society. □

Since we are interested in identifying rules satisfying internal stability, we modify the previous methods by considering only votes to alternatives (S,x) that are internally stable for each $i \in S$ according to R_i; namely, only alternatives (S,x) with the property that $(S,x) P_i [\varnothing]_i$ for each $i \in S$ can receive votes, not only from i but also from all other agents (we call these alternatives *unanimously internal stable*). In approval voting each agent votes, among the set of alternatives at which he is a member of the society, only for those that are unanimously internal stable. If no alternative receives a vote the rule selects a particular alternative $(\varnothing, x)$ by a tie-breaking rule that will be described later. In plurality voting each agent votes for his best unanimously internal stable alternative. In a scoring method we consider only the rank, given agents' preferences, among the unanimously internal stable alternatives.[4] Hence, at the profile of Example 1 each i votes for $[\varnothing]_i$ and $(\varnothing, y)$ is selected according to some preestablished y. With these modifications the three methods satisfy internal stability by definition. Denote by f^P and f^B the plurality voting and the Borda method, respectively.

Our first result is negative: plurality voting and the Borda method do not satisfy consistency (independently of the rule used to break ties).[5] To see that, consider Example 2 below.

Example 2 Let $N = \{1,2,3,4,5,6\}$ and $X = \{y_1, y_2, y_3, y_4, y_5\}$ be the set of agents and outcomes and consider the following $R \in \mathcal{R}$. For each i, $(S,x) P_i [\varnothing]_i$ whenever $i \in S$. In addition, R is one among all profiles satisfying the following properties, where the first column

[4] To obtain the vote of an agent we have to use information contained in the full profile, but since we are not considering the strategic aspect of preference revelation, this is not an issue.

[5] It is easy to see that no scoring method is consistent.

indicates the rank of each of the six preference relations.

Rank	R_1	R_2	R_3	R_4	R_5	R_6
First	(N, y_1)	(N, y_2)	(N, y_3)	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	(N, y_5)
Second	$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	$(N\backslash\{1\}, y_4)$	$(N\backslash\{1\}, y_4)$	
Third	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	$(N\backslash\{2\}, y_4)$	$(N\backslash\{2\}, y_4)$	
Fourth				$(N\backslash\{3\}, y_4)$	$(N\backslash\{3\}, y_4)$	
Fifth				$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	

First, plurality voting does not satisfy consistency since $f^P(R) = (N\backslash\{6\}, y_4)$ but at the same time $f^P\left(R_{|N\backslash\{6\}}\right) = (N\backslash\{6\}, y_1)$. It is possible to select a profile R' satisfying the above rankings in such a way that $f^B(R') = (N\backslash\{6\}, y_4)$ but $f^B(R'_{|N\backslash\{6\}}) = (N\backslash\{6\}, y_1)$. Hence, the Borda method is not consistent. $\square$

Approval voting satisfies not only consistency (and internal stability by definition) but also other desirable properties. Before stating this result we need to specify a tie-breaking rule, to be used whenever two or more alternatives obtain the highest number of votes. Let ρ be a monotonic (strict) order over 2^N. Namely, for each pair $S, T \in 2^N$ such that $S \subsetneq T$, $T\rho S$. Observe that $N\rho S$ for all $S \neq N$.

Fix a monotonic order ρ over 2^N. Denote by $f^{AV,\rho}$ the approval voting that uses ρ to break ties. Formally, let $A' = \{(S_k, x_k)\}_{k=1}^K$ be the set of alternatives that have received the largest number of votes according to approval voting at R. If $S_k = \varnothing$ for all k, select $(\varnothing, y)$ where y is such that $(N, y)P_i(N, z)$ for all $z \neq y$ and $\{i\}\rho\{j\}$ for all $j \neq i$. Assume $S_k \neq \varnothing$ for some k. First select the society $S \in \{S_1, ..., S_K\}$ ranked highest by ρ and consider the subset of alternatives $\{(S_{k'}, x_{k'}) \in A' \mid S_{k'} = S\}$. Select again the agent $i \in S$ who is ranked highest by ρ (as a singleton set) and choose finally as $f^{AV,\rho}(R)$ the alternative most preferred by i among those in the family $\{(S_{k'}, x_{k'}) \in A' \mid S_{k'} = S\}$.

Proposition 1 below states that any approval voting $f^{AV,\rho}$ is internally stable, consistent, and additionally satisfies other desirable properties.

Proposition 1 *Let ρ be a monotonic order over 2^N. Then, the approval voting $f^{AV,\rho}$ satisfies internal stability, consistency, efficiency and neutrality. Moreover, in the subdomain of profiles where the tie-breaking rule is not applied, $f^{A,\rho}$ satisfies anonymity.*

Proof Observe that if (S, x) is approved by i, then $i \in S$. This fact will be repeatedly used in the proof.

- Internal stability. By definition, $f^{A,\rho}$ is internally stable.

- Consistency. Let $R \in \mathcal{R}$ be arbitrary and let $f^{AV,\rho}(R) = (S, x)$ be such that $S \neq \varnothing$. The set of agents approving (S, x) at R coincides with the set of agents approving (S, x) at $R_{|f_N^{A,\rho}(R)}$. Hence, $f^{AV,\rho}(R_{|f_N^{A,\rho}(R)}) = f^{AV,\rho}(R)$ and thus, $f^{AV,\rho}$ is consistent.

- Efficiency. Suppose otherwise; in particular, there must exist $R \in \mathcal{R}$ and $(S, x) \in A$ such that $(S, x) R_i f^{AV,\rho}(R)$ for all $i \in N$ and $(S, x) \neq f^{AV,\rho}(R)$. Assume first that $f_N^{AV,\rho}(R) = \varnothing$, which implies that $f^{AV,\rho}(R)$ did not received any vote and $S \neq \varnothing$. By (P.2), $(S, x) P_j f^{AV,\rho}(R)$ for all $j \in S$ and $(S, x) I_j f^{AV,\rho}(R)$ for all $j \notin S$. But this means that (S, x) received more votes than $f^{AV,\rho}(R)$, a contradiction. Assume now that $f_N^{AV,\rho}(R) \neq \varnothing$ and let $i \in f_N^{AV,\rho}(R)$. Since $f^{AV,\rho}$ satisfies internal stability, $f^{AV,\rho}(R) P_i [\varnothing]_i$. Hence, $i \in S$ and, by the contradiction hypothesis and (P.2), $(S, x) P_i f^{AV,\rho}(R)$. We consider two cases. First, $f_N^{AV,\rho}(R) \subsetneq S$. Since for each $j \in S \backslash f_N^{AV,\rho}(R)$, $f^{AV,\rho}(R) \in [\varnothing]_j$ and $(S, x) R_i f^{AV,\rho}(R)$ for all i, it follows that (S, x) has received more votes than $f^{AV,\rho}(R)$, a contradiction. Second, $f_N^{AV,\rho}(R) = S$. Then, $f^{A,V\rho}(R) = (S, y)$ with $y \neq x$ and all agents in S have approved both, (S, x) and (S, y). This means that the tie-breaking rule ρ has been used to select $f^{AV,\rho}(R)$, implying that there exists $i \in S$ such that $f^{AV,\rho}(R) P_i (S, x)$ which is a contradiction.

- Neutrality. Let $R \in \mathcal{R}$ and $\sigma \in \Sigma$. Observe that the number of agents approving (S, x) at R coincides with the number of agents approving $(S, \sigma(x))$ at R^σ. We consider two cases. First, $f^{AV,\rho}(R)$ has been approved at R by more agents than any other alternative. Hence, $(f_N^{AV,\rho}(R), \sigma(f_X^{AV,\rho}(R)))$ has been approved at R^σ by more agents than any other alternative, implying that $f^{AV,\rho}(R^\sigma) = f^{AV,\rho}(R)$. Second, it is necessary to apply ρ to select $f^{AV,\rho}(R)$. Let $\{(S_k, x_k)\}_{k=1}^K$ be the set of alternatives receiving the largest number of votes at R. Thus, $\{(S_k, \sigma(x_k))\}_{k=1}^K$ is the set of alternatives receiving the largest number of votes at R^σ. Hence,

$$f_N^{AV,\rho}(R^\sigma) = f_N^{AV,\rho}(R). \tag{1}$$

  Now, let $i \in f_N^{AV,\rho}(R)$ be the agent with the highest ranking, among singleton sets, according to ρ and let $i' \in f_N^{AV,\rho}(R^\sigma)$ be the agent with the highest ranking, among singleton sets, according to ρ. By (1) $i' = i$. Thus, $f_X^{AV,\rho}(R^\sigma) = \sigma(f_X^{AV,\rho}(R))$, which together with (1) implies that $f^{AV,\rho}(R^\sigma) = (f_N^{AV,\rho}(R), \sigma(f_X^{AV,\rho}(R)))$.

- Anonymity on the subdomain of profiles where the tie-breaking rule is not applied. Let R be one of such profiles. Then, $f^{AV,\rho}(R)$ has been approved by more agents than any other alternative. Observe that the number of agents approving any (S,x) at R coincides with the number of agents approving $(\pi(S),x)$ at R^{π}. Thus, $(\pi(f_N^{AV,\rho}(R)), f_X^{AV,\rho}(R))$ has been approved at R^{π} by more agents that any other alternative. Hence, $f^{AV,\rho}(R^{\pi}) = (\pi(f^{AV,\rho}(R)), f_X^{AV,\rho}(R))$, which means that $f^{AV,\rho}$ satisfies anonymity at R. ■

We end this note by applying the Condorcet winner to our setting. First, we recall the definition of the Condorcet winner in the classical setting. Fix a profile P of strict preferences over X and let $x, y \in X$ be such that $x \neq y$. We say that x beats y if the number of agents preferring x to y is strictly larger that the number of agents preferring y to x. We say that x is a Condorcet winner at P if there is no y that beats x. There are profiles at which no Condorcet winner exists and others at which there are several Condorcet winners. Thus, the Condorcet winner is not a rule.

We adapt the notion of a Condorcet winner to our setting as we have already did for the previous three rules. In order to ensure that the chosen alternative satisfies internal stability at R we only consider votes for unanimously internal stable alternatives at R. When several Condorcet winners exist we apply the tie-breaking (using a monotonic order ρ) used to define approval voting.

We say that a profile $R \in \mathcal{R}$ is *resolute* if there exists $a \in A$ such that a beats a' for all $a' \neq a$. Thus, the Condorcet winner selects a at R. Let $f^{C,\rho}(R)$ denote the Condorcet winner (if any) at R. If $R \in \mathcal{R}$ is resolute, then $f^{C,\rho}(R)$ is independent of ρ and $\left|f^{C,\rho}(R)\right| = 1$. Proposition 2 states that the Condorcet winner at resolute profiles satisfies the same properties as Approval voting, at such profiles.

Proposition 2 *Let R be a resolute profile. Then, $f^{C,\rho}(R)$ satisfies internal stability, consistency, efficiency, neutrality and anonymity at R.*

Proof Fix a resolute profile R and set $f^{C,\rho}(R) = (S,x)$. We show that $f^{C,\rho}(R)$ satisfies the properties at R.

- Internal stability. By definition, $f^{C,\rho}(R)$ satisfies internal stability at R.

- Consistency. We prove that $f^{C,\rho}\left(R_{|S}\right) = (S,x)$ by showing that at $R_{|S}$, (S,x) beats (T,y) for all $(T,y) \neq (S,x)$ with $T \subset S$. Let (T,y) be an alternative with the above properties. Since (S,x) beats (T,y) at R, the number of agents in N preferring (S,x) to (T,y) is strictly larger than the number of agents in N preferring (T,y) to (S,x). Moreover, each agent in $N \backslash S$ is indifferent between (S,x) and (T,y). Thus the number of agents in S preferring (S,x) to (T,y) (or (T,y) to (S,x)) coincides with the number of agents in N preferring (S,x) to (T,y) (or (T,y) to (S,x)). Hence, (S,x) beats (T,y) at $R_{|S}$, implying that $f^{C,\rho}\left(R_{|S}\right) = (S,x)$.

- Efficiency. Suppose otherwise; in particular, there must exist (T,y) such that $(T,y)\,R_i\,(S,x)$ for all $i \in N$ and $(S,x) \neq (T,y)$. Let $i \in S$. Since (S,x) satisfies internal stability at R, $(S,x)\,P_i\,[\varnothing]_i$. Hence, $i \in T$ and $(T,y)\,P_i\,(S,x)$. Each agent in $N\backslash T$ is indifferent between (S,x) and (T,y). Thus (T,y) beats (S,x), which contradicts that $f^{C,\rho}(R) = (S,x)$.

- Neutrality. Observe that for each $(T,y) \neq (S,x)$, $(S,\sigma(x))$ beats $(T,\sigma(y))$ at R^{σ}. Hence, $f^{C,\rho}(R^{\sigma}) = (S,\sigma(x))$, which means that $f^{C,\rho}$ satisfies neutrality at R.

- Anonymity. Observe that for each $(T,y) \neq (S,x)$, $(\pi(S),x)$ beats $(\pi(T),y)$ at R^{π}. Hence, $f^{C,\rho}(R^{\pi}) = (\pi(S),x)$, which means that $f^{C,\rho}$ satisfies anonymity at R. ■

Nevertheless, for non-resolute profiles the Condorcet winner, even when it is unique, may not satisfy consistency. To see that, consider the following example.

Example 3 Let $N = \{1,2,3,4,5\}$ and $X = \{y_1, y_2\}$ the set of agents and outcomes and let ρ be any monotonic order satisfying $\{1\}\,\rho\,\{2\}\,\rho\,\{3\}\,\rho\,\{4\}\,\rho\,\{5\}$. Consider any profile R satisfying the following properties, where the first column indicates the rank of each of the five preference relations.

Rank	R_1	R_2	R_3	R_4	R_5
First	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$	(N,y_1)
Second	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$	(N,y_1)	(N,y_1)	$[\varnothing]_5$
Third	(N,y_1)	(N,y_1)	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$	
Fourth	$[\varnothing]_1$	$[\varnothing]_2$	$[\varnothing]_3$	$[\varnothing]_4$	

The only internally stable alternatives are $(N\backslash\{5\},y_1)$, $(N\backslash\{5\},y_2)$, and (N,y_1). At R, $(N\backslash\{5\},y_1)$ ties with $(N\backslash\{5\},y_2)$ (so they do not beat each other), $(N\backslash\{5\},y_2)$ beats (N,y_1) and (N,y_1) beats $(N\backslash\{5\},y_1)$. Therefore, R is not resolute because $(N\backslash\{5\},y_2)$ does not beat $(N\backslash\{5\},y_1)$. Since $(N\backslash\{5\},y_2)$ the unique Condorcet winner (no alternative beats it), $f^{C,\rho}(R) = (N\backslash\{5\},y_2)$. To check for consistency of f, consider the subprofile $R_{|N\backslash\{5\}}$ given by

	$(R_{\vert N\backslash\{5\}})_1$	$(R_{\vert N\backslash\{5\}})_2$	$(R_{\vert N\backslash\{5\}})_3$	$(R_{\vert N\backslash\{5\}})_4$
First	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$
Second	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$
Third	$[\varnothing]_1$	$[\varnothing]_2$	$[\varnothing]_3$	$[\varnothing]_4$

At $R_{|N\backslash\{5\}}$, $(N\backslash\{5\}, y_1)$ ties with $(N\backslash\{5\}, y_2)$ and they beat all other alternatives. Hence, the two are Condorcet winners at $R_{|N\backslash\{5\}}$. Thus, applying the tie-breaking rule ρ, and since 1 prefers $(N\backslash\{5\}, y_1)$ to $(N\backslash\{5\}, y_2)$, we have that $f^{C,\rho}\left(R_{|N\backslash\{5\}}\right) = (N\backslash\{5\}, y_1)$, which means that $f^{C,\rho}$ does not satisfy consistency. $\square$

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